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1978 J. Phys. A: Math. Gen. 11 1235

(http://iopscience.iop.org/0305-4470/11/7/011)

J. Phys. A: Math. Gen., Vol. 11, No. 7, 1978. Printed in Great Britain

Exact solutions of Schrödinger's equation for translation-invariant harmonic matter

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Received 21 December 1977, in final form 21 February 1978

Abstract. Earlier exact solutions of Schrödinger's equation for translation-invariant systems of particles interacting by Hooke's law pair potentials are augmented to include systems consisting of an arbitrary number S of *groups* of identical particles. Exact solutions can always be found whenever the number of distinct masses plus the number of distinct coupling constants does not exceed $(2S+1)$. The case $S=2$ is solved in detail and is applied to show that harmonic matter is never stable.

1. Introduction

In an earlier paper (Hall 1978) we considered some exactly soluble *translation-invariant* many-body problems involving Hooke's law pair potentials. By choosing a suitable set of relative coordinates we could solve Schrödinger's equation exactly for: (*a*) a system of N particles with distinct masses and the same coupling constant between all pairs; and (*b*) the 'harmonic atom' system consisting of a particle with mass m_0 and N identical particles with masses m_1, and distinct coupling constants for the identical and non-identical pairs respectively.

In this paper we extend the collection of exactly soluble problems to include systems consisting of S *groups* of identical particles with S distinct coupling constants, one for each group, and a further $\{S+1-n(m)\}$ distinct coupling constants for the group–group pairs, where $n(m)$ is the number of distinct masses. Thus we can allow $(2S+1)$ distinct parameters (masses and coupling constants) in the Hamiltonian of the exactly soluble system. This result could perhaps also be obtained by arguments based on the algebraic problem concerning the simultaneous diagonalisation of two quadratic forms with side conditions (representing the translation-invariance requirements). However, we have employed a purely constructive argument which yields suitable relative coordinates explicitly, together with the eigenfunctions and eigenvalues of the Hamiltonian. Eigenfunctions with particular spatial permutation symmetries may be constructed as sums of single-product eigenfunctions of the Hamiltonian in diagonal form.

Our use in the title of the expression 'harmonic matter' is a reference to the work of Dyson, Lenard and Lieb (Dyson and Lenard 1967, Dyson 1967, Lenard and Dyson 1968, Lieb 1976) on the stability of matter problem which concerns a system of particles interacting by Coulomb pair potentials and having zero total charge. If, for example, there are N positive and N negative particles, Dyson and Lenard have

0305-4770/78/0007-1235$01.00

proved that it is necessary and sufficient for stability (i.e. the ground-state energy E_0 is asymptotically proportional to N) that at least one set of particles be identical fermions. In the present article we solve the analogous problem with Hooke's law pair potentials and we find (§ 3) that harmonic matter is never stable in the above sense. Of course, the analogy is somewhat limited by the fact that ground-state energy does not represent binding energy for the harmonic oscillator.

2. Exact solutions to Schrödinger's equation

We consider first a system of $N_1 \geqslant 1$ identical particles of mass m_1 and $N_2 \geqslant 1$ identical particles of mass m_2 interacting by Hooke's law pair forces (i.e. $S = 2$). The translation-invariant Hamiltonian H for this system is as follows:

$$H = \Big[\sum_{i=1}^{N_1} \frac{\boldsymbol{p}_i^2}{2m_1} + \sum_{i=1}^{N_1} \frac{\boldsymbol{p}_i'}{2m_2} - \Big(\sum_i^{N_1} \boldsymbol{p}_i + \sum_i^{N_2} \boldsymbol{p}_i' \Big)^2 (2m)^{-1} + \sum_{i<j}^{N_1} k_1^2 (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 + \sum_{i<j}^{N_2} k_2^2 (\boldsymbol{r}_i' - \boldsymbol{r}_j')^2 + \sum_i^{N_1} \sum_j^{N_2} k_3^2 (\boldsymbol{r}_i - \boldsymbol{r}_j')^2 \Big], \tag{1}$$

where the total mass $m = N_1 m_1 + N_2 m_2$. In order to solve Schrödinger's equation for this problem we choose a new system of coordinates $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_{N_1}, \boldsymbol{\rho}_2', \boldsymbol{\rho}_3', \ldots, \boldsymbol{\rho}_{N_2}', \boldsymbol{\sigma})$ where $\boldsymbol{\rho}_1$ is the centre-of-mass coordinate and the remaining variables are translation-invariant relative coordinates. We use normalised Jacobi coordinates for each sub-system of identical particles and one additional relative coordinate $\boldsymbol{\sigma}$. If $\boldsymbol{\rho}$ and $\boldsymbol{r}$ are column vectors of the new and old coordinates respectively, then the matrix $\mathbf{B}$ in the coordinate transformation $\boldsymbol{\rho} = \mathbf{B}\boldsymbol{r}$ has the following form:

$$\mathbf{B} = \left[\begin{array}{cccc|cccc} \frac{m_1}{m} & \frac{m_1}{m} & \cdots & \frac{m_1}{m} & \frac{m_2}{m} & \frac{m_2}{m} & \cdots & \frac{m_2}{m} \\ \hline \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \;\; . \;\; 0 & & & & & \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \;\; 0 \;\; . \;\; 0 & & & \bigcirc & & \\ & & \text{etc} & & & & & \\ \hline & & & & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \;\; . \;\; 0 & \\ & \bigcirc & & & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \;\; 0 \;\; . \;\; 0 & \\ & & & & & & \text{etc} & \\ \hline a & a & \cdots & a & -b & -b & \cdots & -b \end{array}\right], \tag{2}$$

where $N_1 a = N_2 b$ and $N_1 a^2 + N_2 b^2 = 1$. In terms of the relative momenta $\boldsymbol{\pi}_i = -i\hbar\nabla_{\boldsymbol{\rho}_i}$,

$\boldsymbol{\pi}_i' = -i\hbar\nabla_{\boldsymbol{\rho}_i'}$, and $\boldsymbol{\pi} = -i\hbar\nabla_{\boldsymbol{\sigma}}$ the operator T representing total *relative* kinetic energy becomes:

$$T = \sum_{i=2}^{N_1} \frac{\boldsymbol{\pi}_i^2}{2m_1} + \sum_{i=2}^{N_2} \frac{(\boldsymbol{\pi}_i')^2}{2m_2} + \frac{1}{N}\left(\frac{N_2}{m_1} + \frac{N_1}{m_2}\right)\boldsymbol{\pi}^2. \tag{3}$$

Rows 2 to N of **B** are a set of orthonormal vectors each with a zero sum of elements. Hence we get (Hall 1978) the following identities for sums of pair-distances squared:

$$\sum_{i<j}^{N_1} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 = N_1 \sum_{i=2}^{N_1} \boldsymbol{\rho}_i^2, \tag{4}$$

$$\sum_{i<j}^{N_2} (\boldsymbol{r}_i' - \boldsymbol{r}_j')^2 = N_2 \sum_{i=2}^{N_2} (\boldsymbol{\rho}_i')^2, \tag{5}$$

$$\sum_{\text{all pairs}} (\text{pair distance})^2 = N\left(\sum_{i=2}^{N_1} \boldsymbol{\rho}_i^2 + \sum_{i=2}^{N_2} (\boldsymbol{\rho}_2')^2 + \boldsymbol{\sigma}^2\right). \tag{6}$$

With these results the translation-invariant Hamiltonian can now be separated completely giving (from equations (1), (3), (4), (5) and (6)):

$$H = \sum_{i=2}^{N_1} \left(\frac{\boldsymbol{\pi}_i^2}{2m_1} + (N_1k_1^2 + N_2k_3^2)\boldsymbol{\rho}_i^2\right) + \sum_{i=2}^{N_2} \left(\frac{(\boldsymbol{\pi}_i')^2}{2m_2} + (N_2k_2^2 + N_1k_3^2)(\boldsymbol{\rho}_i')^2\right)$$
$$+ \left[\frac{1}{N}\left(\frac{N_2}{m_1} + \frac{N_1}{m_2}\right)\boldsymbol{\pi}^2 + Nk_3^2\boldsymbol{\sigma}^2\right]. \tag{7}$$

We note that the diagonalisation leading to equation (7) will go through even if k_1^2 and k_2^2 are *negative* provided that $N_1k_1^2 + N_2k_3^2 > 0$ and $N_2k_2^2 + N_1k_3^2 > 0$. The unsymmetrised eigenfunctions of H are products of $3(N-1)$ Hermite functions and the eigenvalues are the corresponding sums of single-particle energies. The burden of spatial permutation symmetry is carried for the N_1 particles entirely by the dependence of the state on the $\boldsymbol{\rho}$ variables, and for the N_2 particles by the dependence of the state on the $\boldsymbol{\rho}'$ variables; $\boldsymbol{\sigma}$ is symmetric in $(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{N_1})$ and also in $(\boldsymbol{r}_1', \boldsymbol{r}_2', \ldots, \boldsymbol{r}_{N_2}')$. Equation (7) is consistent with the solution we found (Hall 1978) to the 'harmonic atom' system in which $N_2 = 1$.

For the special case of *scalar* particles in one spatial dimension we find from equation (7) (Post 1953, Hall 1978) the following expression for the ground-state energy E_0:

$$E_0 = A(N_1)\hbar\left(\frac{N_1k_1^2 + N_2k_3^2}{2m_1}\right)^{1/2} + A(N_2)\hbar\left(\frac{N_2k_2^2 + N_1k_3^2}{2m_2}\right)^{1/2} + a\hbar k_3\left(\frac{N_2}{m_1} + \frac{N_1}{m_2}\right)^{1/2}, \tag{8}$$

where for *bosons* $A(N) = (N-1)$ and for *fermions* $A(N) = (N^2-1)$; in both cases $a = 1$.

In three dimensions the ground-state energy when both groups are bosons is given by the expression in equation (8) multiplied by the factor three. The corresponding formula for (scalar) fermions in three dimensions is more complicated because we must anti-symmetrise in the individual particle indices for each set of identical fermions and allow for all possible linearly independent states (Post 1953, Hall 1978). There are successive shells of $\frac{1}{2}q(q+1)$ states, $q = 1, 2, 3, \ldots$ and, in order to obtain a simple formula, we choose the number of fermions so that exactly ν shells are filled.

For example if the N_1 particles are fermions we have:

$$N_1 = \sum_{q=1}^{\nu_1} \tfrac{1}{2}q(q+1) = \tfrac{1}{6}\nu_1(\nu_1^2+3\nu_1+2) \equiv f(\nu_1). \tag{9}$$

The successive energies are proportional to $(2q+1)$ and consequently the factors $A(N_1)$ in equation (8) must be replaced (Post 1953) by

$$g(\nu_1) = \sum_{q=1}^{\nu_1} \tfrac{1}{2}q(q+1)(2q+1) - 3 = \tfrac{1}{4}\nu_1(\nu_1^3+4\nu_1^2+5\nu_1+2) - 3. \tag{10}$$

Hence in three dimensions the ground-state energy (for scalar particles) is given by equation (8) in which $a = 3$ and we have:

bosons: $$A(N_i) = 3(N_i - 1), \qquad i = 1 \text{ or } 2. \tag{11}$$

fermions: $$\left.\begin{aligned} N_i &= f(\nu_i) \\ A(N_i) &= g(\nu_i) \end{aligned}\right\} \qquad i = 1 \text{ or } 2. \tag{12}$$

The most general problem of this type that we are able to solve exactly consists of S *groups* of identical particles with particle numbers $(N_1, N_2, \ldots, N_S)$ and masses $(m_1, m_2, \ldots, m_S)$. For this general problem we take the transformation matrix $\mathbf{B}$ to be the natural generalisation of equation (2) with S 'Jacobi blocks' and $(S-1)$ $\boldsymbol{\sigma}$-variables, $(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \ldots, \boldsymbol{\sigma}_S)$, defined by the last $(S-1)$ rows of $\mathbf{B}$, which we choose to have the following form:

$$\left[\begin{array}{l|l|l|l|l} a_2 a_2 \ldots a_2 & -b_2 -b_2 \ldots -b_2 & 0 \quad \ldots \quad 0 & 0 \ldots 0 & \ldots \\ a_3 a_3 \ldots a_3 & a_3 \; a_3 \ldots \; a_3 & -b_3 -b_3 \ldots -b_3 & 0 \ldots 0 & \ldots \\ & \text{etc} & & & \end{array}\right] \tag{13}$$

Hence, in addition to the S identities like equation (4), there will be a further $(S-1)$ identities like equation (6) and evidently we could admit at least $(S-1)$ distinct group–group coupling constants. However there is a snag in this argument. If the particle masses $(m_1, m_2, \ldots, m_S)$ are all *distinct*, then choosing the last $(S-1)$ rows of $\mathbf{B}$ to have the form (13) will *not* allow us to write the relative kinetic energy T in diagonal form: there will now be cross-terms between the momenta associated with the $\boldsymbol{\sigma}$-variables. In this extreme case of S distinct masses, we could still solve the problem exactly if the coupling constants between *all* pairs of *different* particles are equal (i.e. the extension of Hall 1978, §2 to *groups* of identical particles): we should then continue $\mathbf{B}$ *not* as in the array (13) but rather as a generalisation of the matrix $\mathbf{B}$ in Hall (1978, §2); and use only *one* identity like equation (6).

The situation for S groups of identical particles (with the same coupling constant between all pairs from the same group) can be summarised as follows. We can solve the problem exactly if the number of distinct masses $n(m)$ and the number of distinct group–group coupling constants $n(k)$ satisfy

$$n(m) + n(k) = S + 1. \tag{14}$$

Of course, we also have, in addition, the S coupling constants for pairs within each given group of identical particles (for the present argument we assume $N_i \geqslant 2$, $i = 1, 2, \ldots, S$). Thus the exactly soluble problem can have $(2S+1)$ distinct parameters corresponding to the $(2S+1)$ independent parameters which appear in the diagonal form of H (for example, equation (7) in which $S = 2$).

As we have observed earlier (Hall 1978, § 5) we could actually allow up to N_i distinct coupling constants (instead of 1) *within* each group of $N_i > 2$ identical particles. In this event we should have altogether $\Sigma_i^S N_i + (S+1) = (N+S+1)$ distinct parameters allowed. However, since the particles in each group are identical in the present problem, it does not make much physical sense to favour some pairs within the same group with a stronger coupling than others.

In the case of N particles with N *distinct* masses (or the corresponding problem with groups of particles) it may still be possible to obtain exact solutions and allow some choice of different coupling constants. Our constructive method has not yielded a solution in this case and cannot, of course, be used to rule out the possibility. If one removes the mass distinction by an initial scale change then the construction of translation-invariant coordinates becomes rather complicated. This is why we have always used translation-invariant coordinates from the outset.

3. Conclusion

In this paper and an earlier article (Hall 1978) we have established a large class of exactly soluble translation-invariant many-body problems involving Hooke's law pair potentials. We have used a set of normalised Jacobi relative coordinates for each group of identical particles in the system so that we can apply the analysis of Post (1953) to construct energy eigenstates which satisfy the Pauli principle for those groups which are fermions. The remaining relative coordinates are then chosen to be symmetric under the permutation of identical particles and such that the maximum number of distinct group–group coupling constants is accommodated whilst the kinetic energy operator remains diagonal.

As an application of the solutions obtained in § 2 we consider a system consisting of two groups each of N identical particles in three dimensions and we arrange that 'unlike particles attract' and 'like particles repel' by setting $\lambda k_1^2 = \lambda k_2^2 = k_3^2 = k^2 > 0$, and $-1 < \lambda < 0$. We then obtain the formal analogue of the 'stability of matter problem' considered by Dyson and Lenard (1967), but with harmonic potentials replacing Coulomb potentials. From equations (8), (11) and (12) in § 2 above we obtain in this case the following results:

both groups bosons:

$$E_0 \sim N^{3/2}$$

at least one group fermions:

$$E_0 \sim N^{11/6},$$

where '$\sim$' indicates the asymptotic form of N dependence for large N. We get the same forms of N dependence of E_0 if we consider more than two groups of particles with fixed ratios of particle numbers and coupling constants. A 'stable system' (with $E_0 \sim N$) of this type is therefore impossible†. The general energy lower-bound methods for many-body problems (Hall and Post 1967, Hall 1967, Carr and Post 1977) which have (to some extent) been developed from exact harmonic oscillator solutions have, with one pathological exception (Horton 1973), generally given disappointing results when applied to stable or saturating systems in which $E_0 \sim N$. A

† However, a specially constructed N-dependent λ of the form $\lambda = -(1 - N^{-5/3})$ *would* yield $E_0 \sim N$.

goal for these energy lower-bound studies might be to develop a general method which would provide in particular a short proof of the Dyson and Lenard theorem (Dyson and Lenard 1967, Lieb 1976) which states that $E_0 \sim N$ for the above $2N$-body problem with Coulomb pair potentials, provided at least one group of particles consists of identical fermions.

Although the number of quarks required by elementary-particle theory is increasing at an alarming rate (Harari 1976) a sufficiently large variety of exactly soluble quark models is now possible with the solutions we have found in this paper and in Hall (1978). For example, we could now consider a four-quark model with two distinct masses ($S = 2$) and three different coupling constants; or even a six-quark model with, say, three distinct masses ($S = 3$) and four different coupling constants. Of course, the very tight binding provided by the harmonic oscillator potential is appropriate for systems of quarks.

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